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ON A CLASS OF RANK ORDER TESTS FOR INDEPENDENCE IN MULTIVARIATE DISTRIBUTIONS

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1. Summary. In this paper we offer nonparametric competitors of some classical tests of multidimensional independence considered by Wilks (1935), Daly (1940) and, Wald and Brookner (1941). In this context, the problems of testing (i) the mutual independence of q subsets of the totality of p variates ($p \geq q \geq 2$), and (ii) pair wise independence of p variates [cf. Anderson (1957) Chapter 9] are considered, all against appropriate classes of stochastic dependence alternatives. Some strictly distribution free permutation tests are developed here, and their asymptotic properties are studied with the aid of a theorem on the asymptotic distribution of a class of rank order statistics to the case of more than two variates.

2. Introduction. Let $\underset{\sim}{X}_\alpha = (X_{1\alpha}, \ldots, X_{p\alpha})$, $\alpha=1,\ldots,n$ be n i.i.d.r.v's (independent and identically distributed (vector valued) random variables) having a $p(\geq 2)$ variate continuous cdf (cumulative distribution function) $F(\underset{\sim}{x})$, where $\underset{\sim}{x} = (x_1, \ldots, x_p) \in R^p$, the p-dimensional Euclidean space. Let $\underset{\sim}{X}_\alpha$ be partitioned into $q(\geq 2)$ subvectors; that is

$$\underset{\sim}{X}_\alpha = (\underset{\sim}{X}_\alpha^{(1)}, \ldots, \underset{\sim}{X}_\alpha^{(q)}); \quad \alpha=1,\ldots,n \tag{2.1}$$

where $\underset{\sim}{X}_{\alpha}^{(j)}$ is of i_j components; $j=1,\ldots,q$ and $\sum_{j=1}^{q} i_j = p$. Let the marginal cdf of $\underset{\sim}{X}_{\alpha}^{(j)}$ be denoted by $F^{(j)}(\underset{\sim}{x}^{(j)})$ where $\underset{\sim}{x}^{(j)} \in R^{i_j}$, $j=1,\ldots,q$. Our first problem relates to the stochastic independence of the q subsets of variates. This may be statistically framed by means of the null hypothesis

$$H_o^{(q)}: \quad F(\underset{\sim}{x}) = \prod_{j=1}^{q} F^{(j)}(\underset{\sim}{x}^{(j)}) \quad \text{for all } \underset{\sim}{x} \in R^p. \tag{2.2}$$

Let us denote the marginal cdf of $X_{i\alpha}$ by $F_{[i]}(x)$, $i=1,\ldots,p$; and the marginal joint cdf of $(X_{i\alpha}, X_{j\alpha})$ by $F_{[i,j]}(x,y)$ for all $i \neq j=1,\ldots,p$. $X_{1\alpha},\ldots,X_{p\alpha}$ are said to be totally independent if

$$H_o^{(p)}: \quad F(\underset{\sim}{x}) = \prod_{i=1}^{p} F_{[i]}(x) \quad \text{for all } \underset{\sim}{x} \in R^p . \tag{2.3}$$

Thus $H_o^{(p)}$ is a particular case of $H_o^{(q)}$ when $q=p$, that is when, $i_1=\ldots,i_p=1$. Similarly, $X_{1\alpha},\ldots,X_{p\alpha}$ are said to be pair wise independent if

$$H_o^{(*)}: \quad F_{[i,j]}(x,y) = F_{[i]}(x)\, F_{[j]}(y) \quad \text{for } (x,y) \in R^2 \quad , \tag{2.4}$$

and for all $i \neq j=1,\ldots,p$. (It may be noted that (2.3) implies (2.4) but not the vice-versa).

In the particular case of $F(\underset{\sim}{x})$ being a multinormal cdf, pair wise independence implies total independence and vice-versa. Moreover in this case uncorrelation implies independence and vice-versa. Also, the multinormal cdf is completely specified by its mean vector and the covariance matrix. Thus, all the three problems stated above reduce to certain specific structures of the covariance matrix. If the covariance matrix is partitioned into q^2 submatrices of orders $i_j \times i_\ell$; $j,\ell=1,\ldots,q$, then the first hypothesis $H_o^{(q)}$ is equivalent to the

hypothesis that all the off-diagonal partitioned matrices are the null matrices, while the hypotheses $H_o^{(p)}$ and H_o^* are both equivalent to the hypothesis that the covariance matrix itself is a diagonal matrix. Tests for these hypotheses are considered in Anderson (1957, Chapter 9), and the reader is referred to it for details of background and motivation.

The object of the present investigation is to explore the possibility of generalizing the above problems in a completely nonparametric set up. Now, for arbitrary (continuous) multivariate distributions the covariance matrix may not exist, and even if it exists, it may not play the fundamental role as in the case of the multinormal distributions. For this reason, we shall formulate a class of association parameters which are regular functionals of the cdf $F(\underset{\sim}{x})$ and which are defined for a much wider class of cdf's. This will provide us with a suitable nonparametric competitors of the classical covariance (or correlation) matrix and also increase the scope of the methods. Secondly, for multinormal cdf's uncorrelation and independence are equivalent, but the same is not true for arbitrary cdf's. Thus, to be precise about the class of alternatives, we will also define some dependence function with some emphasis on association alternatives. The proposed tests will be shown to be consistent and reasonably efficient for such alternatives. Finally, for arbitrary cdf's, since pairwise independence does not inply total independence (cf. Geisser and Mantel (1962)), we shall also consider the problem of testing H_o^*, defined by (2.4), in some detail. In this context, we shall extend a theorem by Bhuchongkul (1964) on the asymptotic distribution of (association) rank order statistics to the multivariate case, and subsequently use it to study the asymptotic properties of the proposed tests.

3. Formulation of a class of association parameters. Apart from the necessity of the existence of the second order moments, the estimator of the usual covariance matrix are quite sensitive to outlying observations. Here we shall consider some

alternative measures of association which are really functionals of the parent cdf's. For this, we consider the marginal cdf's $F_{[i]}$ and $F_{[i,j]}$ $(i \neq j=1,\ldots,p)$ as in section 2. Also, let $J_{(i)}(u)$, $0 < u < 1$ $(i=1,\ldots,p)$ be some absolutely continuous function defined on the open interval $(0,1)$ and suppose that $J_{(i)}(u)$ is normalized in the following manner:

$$(3.1) \qquad \int_0^1 J_{(i)}(u)\,du = 0 \quad \text{and} \quad \int_0^1 J^2_{(i)}(u)\,du = 1 \quad \text{for } i=1,\ldots,p.$$

Later on we shall impose certain regularity conditions on $J_{(i)}(u)$. Let us now consider the transformed variables

$$(3.2) \qquad Y_i = J_{(i)}(F_{[i]}(x_i)) \qquad \text{for } i=1,\ldots,p$$

and denote

$$(3.3) \qquad \underset{\sim}{Y} = (Y_1,\ldots,Y_p)$$

Y_i will be called the <u>grade functional</u> of X_i: $i=1,\ldots,p$. The joint distribution of $\underset{\sim}{Y}$ can be obtained from $F(\underset{\sim}{x})$, but it will naturally depend upon the association pattern of $F(x)$. If $J_{(i)}(u)$ is not constant for all $0 < u < 1$, $(i=1,\ldots,p)$ it is easy to verify that independence of (X_i, X_j) implies the independence of (Y_i, Y_j) and vice-versa. Further this property is preserved under any monotone transformation of X_i; $i=1,\ldots,p$. So when we are interested in the problems of stochastic independence we may work with $\underset{\sim}{Y}_\alpha$, $\alpha=1,\ldots,n$, and suitably choosen $J_{(i)}$; $i=1,\ldots,p$, often lead to some nice properties of such alternative measures. Let us now define

$$(3.4) \qquad \theta_{ij} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} J_{(i)}(F_{[i]}(x))\, J_{(j)}(F_{[j]}(y))\, dF_{[i,j]}(x,y); \quad i,j=1,\ldots,p$$

It may be noted that by (3.1), $\theta_{ii} = 1$ for all $i=1,\ldots,p$ and $|\theta_{ij}| \leqq 1$ for all $i \neq j=1,\ldots,p$. Further, if $X_{i\alpha}$ and $X_{j\alpha}$ are independent, then from (3.1) and (3.4), we see that $\theta_{ij} = 0$. In fact θ_{ij} is the product moment correlation of $Y_{i\alpha}$ and $Y_{j\alpha}$. Thus, we define a <u>nonparametric association matrix</u> by

$$(3.5) \qquad \underset{\sim}{\textcircled{H}} = ((\theta_{ij}))_{i,j=1,\ldots,p} .$$

Naturally, the choice of the transformations $J_{(i)}$, $i=1,\ldots,p$ plays an important role in the procedure to be considered. We shall consider a general class of such functionals and subsequently we shall briefly consider the problem of selecting some particular measures.

Now, usually the cdf's $F_{[i]}$, $i=1,\ldots,p$ are all unknown, and hence $\underset{\sim}{Y}$ is also unknown. However, (3.4) has a close analogy with the type of functionals considered by Von Mises (1947), and following his line of approach we shall frame suitable estimators of $\underset{\sim}{\textcircled{H}}$. With this end in view, let us define

$$(3.6) \qquad F_{n[i]}(x) = (\text{number of } X_{i\alpha} \leqq x \text{ , } \alpha=1,\ldots,n)/n; \; i=1,\ldots,p$$

$$(3.7) \qquad F_{n[i,j]}(x,y) = (\text{number of } (X_{i\alpha}, X_{j\alpha}) \leqq (x,y), \; \alpha=1,\ldots,n)/n; \; i \neq j=1,\ldots,p.$$

Then we may consider $J_{(i)}(F_{n[i]}(x))$ as a natural estimator of $J_{(i)}(F_{[i]}(x))$; $i=1,\ldots,p$. Further in actual practice, often we have a sequence of functions viz. $\{J_{n(i)}(u)\}$ which converges to $J_{(i)}(u)$ for all $0 < u < 1$ as $n \rightarrow \infty$, and $i=1,\ldots,p$. Thus for the sake of more generality in presentation, we may consider the estimators

$$(3.8) \qquad J_{n(i)}(F_{n[i]}(x)) .$$

Consequently, from (3.4), (3.6), (3.7) and (3.8), we may frame the estimator of θ_{ij} as $T_{n,ij}$ where

$$(3.9)\qquad T_{n,ij} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} J_{n(i)}(F_{n[i]}(x))\, J_{n(j)}(F_{n[j]}(y))\, dF_{n[i,j]}(x,y)$$

$$= \frac{1}{n}\sum_{\alpha=1}^{n} J_{n(i)}(F_{n[i]}(x_{i\alpha})\, J_{n(j)}(F_{n[j]}(x_{j\alpha})); \; i,j=1,\ldots,p.$$

It may be noted that $T_{n,ii}$ are nonstochastic (if $J_{n(i)}$'s are so) and by (3.1), they converge to unity as $n\to\infty$. Our proposed tests are based on suitable functions of the stochastic matrix $\underset{\sim}{T}_n$, when

$$(3.10)\qquad \underset{\sim}{T}_n = ((T_{n,ij}))_{i,j=1,\ldots,p}\,.$$

In the sequel, $\underset{\sim}{T}_n$ and $\underset{\sim}{\Theta}$ will be termed respectively the sample and population dispersion matrices. It may be noted that many well known nonparametric measures of association belong to the class considered above. For example, if we let

$$(3.11)\qquad J_{n(i)}(\alpha/(n+1)) = \left(\frac{12}{n^2-1}\right)^{\frac{1}{2}} \{\alpha - \frac{n+1}{2}\}; \quad \alpha=1,\ldots, n: \; i=1,\ldots,p$$

then $T_{n,ij}$ reduces to Spearman's rank correlation (cf. Kendall (1955)) while for $J_{(i)}(u) = \sqrt{12}\,(u-\frac{1}{2})$, $0 < u < 1$; $i=1,\ldots,p$ θ_{ij} reduces to the grade correlation coefficient (cf. Hoeffding (1948), pp. 318). The measures considered by Blomquist (1950) and Bhuchongkul (1964) among others, also belong to this class.

We have so far tried to provide a heuristic argument for measuring association by the functionals θ_{ij}'s defined by (3.4). We may also justify the use of such functionals if we adopt Hájek's approach (1962). However, his study specifically relates to linear regression alternatives whereas we are more interested in association alternatives, and apparently his procedure seems to

have very little impact on ours. But, in spite of his logic, one may propose some measures of association which are particular cases of the general class of $\underset{\sim}{\textcircled{H}}$.

Again, in multinormal distributions any stochastic association is reflected by the correlation matrix. In general, deviation from independence may not be reflected by correlation. Thus to consider the class of alternative hypotheses, we will also consider some dependence function (cf. Sibuya (1959)).

Denote

$$\Omega(\underset{\sim}{x}) = F(\underset{\sim}{x}) / \prod_{i=1}^{p} F_{[i]}(x_i) \tag{3.12}$$

$$\Omega_{ij}(x, y) = F_{[i,j]}(x, y) / F_{[i]}(x_i)\, F_{[j]}(y), \quad i \neq j = 1, \ldots, p. \tag{3.13}$$

Both Ω and Ω_{ij} are functions of the marginal distributions as well as the joint ones. So we shall prefer to write them as $\Omega(F_{[1]}, \ldots, F_{[p]})$ and $\Omega_{ij}(F_{[i]}, F_{[j]})$ respectively. Hereafter we shall term Ω and Ω_{ij} as dependence functions. (It may be noted that if $X_1, \ldots, X_p$ are totally independent (or if X_i and X_j are independent), then Ω(or Ω_{ij}) will be equal to unity of all $\underset{\sim}{x}$ (or all x_i, x_j) and so, the divergence from unity on a set of points of measure non-zero relates to stochastic dependence). In subsequent sections, we shall make repeated use of this type of dependence functions.

4. Permutationally distribution-free tests for $H_o^{(q)}$. It may be noted that if X_i and X_j belong to two different subsets, then under $H_o^{(q)}$ in (2.2), $\theta_{ij} = 0$. Let us now denote by $\mathfrak{F}_r$ the class of all r-variate continuous cdf's for r=1,...,p; so that $F(\underset{\sim}{x}) \in \mathfrak{F}_p$.

Let us define

$$(4.1) \qquad \mathfrak{F}_p^o = \{F(\underset{\sim}{x}) : F(\underset{\sim}{x}) = \prod_{j=1}^{q} F^{(j)}(\underset{\sim}{x}^{(j)}), \quad F^{(j)} \in \mathfrak{F}_{i_j}, \quad j=1,\dots,q\}$$

Thus $\mathfrak{F}_p^o$ is a subclass of all p-variate continuous cdf's for which the q specified subsets of variates are mutually stochastically independent. Thus, one may write equivalently

$$(4.2) \qquad H_o^{(q)}: \quad F(\underset{\sim}{x}) \in \mathfrak{F}_p^{(o)}$$

and may be interested in the set of alternatives that $F(\underset{\sim}{x}) \in \mathfrak{F}_p - \mathfrak{F}_p^o$. However, such a formulation will give rise to considerable amount of mathematical complications since $F(\underset{\sim}{x})$ is otherwise of completely unknown form. In this paper, we shall restrict ourselves to the class of alternatives which relate to the lack of independence through the values of the functional matrix $\underset{\sim}{H}$ defined in (3.5). Let us now partition $\underset{\sim}{H}$ as

$$\underset{\sim}{H} = \begin{pmatrix} \underset{\sim}{H}_{11} & \underset{\sim}{H}_{12} & \cdots & \underset{\sim}{H}_{1q} \\ \underset{\sim}{H}_{21} & \underset{\sim}{H}_{22} & \cdots & \underset{\sim}{H}_{2q} \\ \vdots & \vdots & & \vdots \\ \underset{\sim}{H}_{q1} & \underset{\sim}{H}_{q2} & & \underset{\sim}{H}_{qq} \end{pmatrix}$$

where $\underset{\sim}{H}_{k\ell}$ is of the order $i_k \times i_\ell$; $k,\ell=1,\dots,q$. Now from (4.2), we have

$$(4.4) \qquad \underset{\sim}{H}_{k\ell} = \underset{\sim}{0} \quad \text{for} \quad k \neq \ell = 1,\dots,q \quad \text{under } H_o^{(q)}.$$

We frame the class of alternatives as the set of all p-variate cdf's for which the equality sign in (4.4) does not hold for at least one pair (k,ℓ), $k \neq \ell = 1,\dots,q$.

Now recalling the definition of θ_{ij} in (3.4) and Ω_{ij} in (3.13), we may write

$$(4.5)\quad \theta_{ij} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [F_{[i,j]}(x,y) - F_{[i]}(x)\, F_{[j]}(y)]\, J'_{(i)}[F_{[i]}(x)]\, J'_{(j)}[F_{[j]}(y)]\, dF_{[i]}(x)\, dF_{[j]}(y)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_{[i]}(x)\, F_{[j]}(y)\, [\Omega_{ij}(F_{[i]}(x), F_{[j]}(y))-1]\, J'_{(i)}[F_{[i]}(x)]\, J'_{(j)}[F_{[j]}(y)]\, dF_{[i]}(x)\, dF_{[j]}(y)$$

(Here J' denotes the first derivative of J).

In most of the cases (as we shall see later on) $J_{(i)}(u)$ is monotone in u, $0 < u < 1$ for $i=1,\ldots,p$, so that if $\Omega_{ij}(x,y)$ is uniformly (in x,y) greater than (less than) or equal to 1 with the strict inequality on at least a set of points of measure nonzero, then $\theta_{ij} \neq 0$. In many types of factorial dependence (cf. Konijn (1956), Bhuchongkul (1964) for the bivariate case), it is easy to verify that Ω_{ij} is greater than or less than one, depending upon the coefficients of the stochastic factors, and hence such an alternative may be quite suitable. Incidentally, we are not restricting ourselves to the particular type of the alternatives of linear dependence as considered by Konijn (1956) and Bhuchongkul (1964) (in the particular case of bivariate distributions), in fact, they can be shown to be contained in the class of alternatives considered in this paper.

Now for the simplicity of presentation, we shall consider in detail the tests for $H_o^{(q)}$ when $q=2$, and then we shall indicate briefly how the theory can be extended to the case of more than two subsets. Let then

$$(4.6)\quad \underset{\sim}{X}_\alpha = (\underset{\sim}{X}_\alpha^{(1)}, \underset{\sim}{X}_\alpha^{(2)});\quad \underset{\sim}{X}_\alpha^{(1)} = (X_{1\alpha},\ldots,X_{\ell\alpha});\quad \underset{\sim}{X}_\alpha^{(2)} = (X_{\ell+1,\alpha},\ldots,X_{p\alpha})$$

where $1 \leqq \ell$, $m=p-\ell < p$, and $\alpha=1,\ldots,n$.

We denote the sample point E_n by

$$(4.7) \qquad \underset{\sim}{E}_n^{p\times n} = (\underset{\sim}{X}_1', \ldots, \underset{\sim}{X}_n') = \begin{pmatrix} \underset{\sim}{X}_1^{(1)'}, \ldots, \underset{\sim}{X}_n^{(1)'} \\ \underset{\sim}{X}_1^{(2)'}, \ldots, \underset{\sim}{X}_n^{(2)'} \end{pmatrix}$$

The joint distribution function of $\underset{\sim}{E}_n^{p\times n}$ is given by

$$(4.8) \qquad G(\underset{\sim}{E}_n) = \prod_{\alpha=1}^{n} F(\underset{\sim}{x}_\alpha)$$

and by (2.2), under $H_o^{(2)}$

$$(4.9) \qquad G(\underset{\sim}{E}_n) = \prod_{\alpha=1}^{n} \{F^{(1)}(\underset{\sim}{x}_\alpha^{(1)})\ F^{(2)}(\underset{\sim}{x}_\alpha^{(2)})\} .$$

Let now $\underset{\sim}{R}_n = (R_1, R_2, \ldots, R_n)$ be any permutation of $(1,2,\ldots,n)$ and we denote by $\underset{\sim}{\mathcal{R}}_n$, the set of all the $n!$ permutations of $(1,2,\ldots,n)$. Furthermore, let us also denote

$$(4.10) \qquad \underset{\sim}{E}(\underset{\sim}{R}_n) = \begin{pmatrix} \underset{\sim}{X}_1^{(1)'}, \ldots, \underset{\sim}{X}_n^{(1)'} \\ \underset{\sim}{X}_{R_1}^{(2)'}, \ldots, \underset{\sim}{X}_{R_n}^{(2)'} \end{pmatrix} , \quad \underset{\sim}{R}_n \in \underset{\sim}{\mathcal{R}}_n$$

$$(4.11) \qquad S(\underset{\sim}{E}_n) = \{\underset{\sim}{E}(\underset{\sim}{R}_n) : \underset{\sim}{R}_n \in \underset{\sim}{\mathcal{R}}_n\} .$$

Then, it follows from (4.9) and (4.10) that the joint distribution of $\underset{\sim}{E}(\underset{\sim}{R}_n)$ remains invariant under the group of transformations $\underset{\sim}{R}_n \in \underset{\sim}{\mathcal{R}}_n$ if $H_o^{(2)}$ holds.

Consequently, we shall call $S(\underset{\sim}{E}_n)$, <u>the permutation invariant set</u> of $\underset{\sim}{E}_n$. Let now $\underset{\sim}{\mathcal{E}}_n$ be the n p dimensional Euclidean space, so that $\underset{\sim}{E}_n \in \underset{\sim}{\mathcal{E}}_n$. Now under $H_o^{(2)}$, the probability distribution of $\underset{\sim}{E}_n$ over the set of n! points in $S(\underset{\sim}{E}_n)$ is uniform, and hence, if $\phi(\underset{\sim}{E}_n)$ is any test function choosen in such a way that

$$(4.12) \qquad \sum_{\underset{\sim}{E}^*_n \in S(\underset{\sim}{E}_n)} \phi(\underset{\sim}{E}^*_n) = n!\,\epsilon \; ; \quad 0 < \epsilon < 1$$

for all $\underset{\sim}{E}_n \in \underset{\sim}{\mathcal{E}}_n$, then

$$(4.13) \qquad E_{H_o^{(2)}}\{\phi(\underset{\sim}{E}_n)\} = \epsilon, \quad \text{the level of signification.}$$

($E_{H_o^{(q)}}\{\cdot\}$ denotes the expectation of $\{\cdot\}$ under $H_o^{(q)}$).

Now to formulate $\phi(\underset{\sim}{E}_n)$ in a suitable manner and to make it invariant under monotone transformation of the variables, we proceed as follows. For each integer n, let

$$(4.14) \qquad \underset{\sim}{L}_n^{(i)} = (L_{n,1}^{(i)}, \ldots, L_{n,n}^{(i)}); \quad i=1,\ldots,p.$$

be the p sets of real constants, where

$$(4.15) \qquad L_{n,\alpha}^{(i)} = J_n^{(i)}\left(\frac{\alpha}{n}\right); \quad \alpha=1,\ldots,n; \quad i=1,\ldots,p.$$

and where the functions $J_n^{(i)}$ have the same interpretation as in section 3. With this notation we can rewrite $T_{n,ij}$ (cf. (3.9)) as

$$(4.16) \qquad T_{n,ij} = \frac{1}{n} \sum_{\alpha=1}^{n} L_{n,R_{i\alpha}}^{(i)} \; L_{n,R_{j\alpha}}^{(j)} \; ; \qquad i,j=1,\ldots,p$$

where $R_{i\alpha}$ is the rank of $X_{i\alpha}$ among $(X_{i1},\ldots,X_{in})$ for $i=1,\ldots,n$. Under $H_o^{(2)}$, $X_{i\alpha}$ and $X_{j\alpha}$ are stochastically independent for all $i=1,\ldots,\ell$ and $j=\ell+1,\ldots,p$.

Let us now partition $\underset{\sim}{T}_n = ((T_{n,ij}))$ $i,j=1,\ldots,p$ as we did $\underset{\sim}{H}$ in (4.3) for $q=2$. We then write

$$\underset{\sim}{T}_n = \begin{pmatrix} ((T_{n,ij}^{(1,1)})) & ((T_{n,ij}^{(1,2)})) \\ ((T_{n,ij}^{(2,1)})) & ((T_{n,ij}^{(2,2)})) \end{pmatrix} \tag{4.17}$$

note that

$$\begin{aligned} &((T_{n,ij}^{(1,1)})) = ((T_{n,ij}))\ i,j=1,\ldots,\ell\ ; \quad ((T_{n,ij}^{(1,2)})) = ((T_{n,ij}))\ \begin{matrix} i=1,\ldots,\ell \\ j=\ell+1,\ldots,p \end{matrix} \\ &((T_{n,ij}^{(2,1)})) = ((T_{n,ij}))\ \begin{matrix} i=\ell+1,\ldots,p \\ j=1,\ldots,\ell \end{matrix}\ ; \quad ((T_{n,ij}^{(2,2)})) = ((T_{n,ij}))\ \begin{matrix} i=\ell+1,\ldots,p \\ j=\ell+1,\ldots,p \end{matrix}\ . \end{aligned} \tag{4.18}$$

Let us now denote by $\mathcal{P}_n$, the permutational probability law generated by the n! equally likely (conditional) realizations of $\underset{\sim}{E}_n$ over $S(\underset{\sim}{E}_n)$. It is then easily seen that

$$E(T_{n,ij} \mid \mathcal{P}_n) = 0 \quad \text{for all } i=1,\ldots,\ell\ ; \text{ and } j=\ell+1,\ldots,p. \tag{4.19}$$

$$\operatorname{Cov}(T_{n,ij}, T_{n,i'j'} \mid \mathcal{P}_n) = \frac{1}{n-1} T_{n,ii'} T_{n,jj'} \tag{4.20}$$

for all $i,i'=1,\ldots,\ell$ and $j,j'=\ell+1,\ldots,p$.

Since we are interested in a comprehensive test for $H_o^{(2)}$ (i.e., when $q=2$), following some simple steps we may consider the test statistic

$$V_{n(2)}^{(J)} = (n-1) \sum_{i=1}^{\ell} \sum_{i'=1}^{\ell} \sum_{j=\ell+1}^{p} \sum_{j'=\ell+1}^{p} T_{n,ij} T_{n,i'j'} T_n^{ii'} T_n^{jj'} \tag{4.21}$$

where $T^{ii'}_{n(1)}$ is the (i,i')th term in $((T^{(1,1)}_{n,ij}))^{-1}$ and $T^{jj'}_{n(2)}$ is the (j,j')th term in $((T^{(2,2)}_{n,ij}))^{-1}$. However, we shall find it more convenient to work with the test-statistic $S_{n(2)}$ defined as

$$(4.22) \qquad S^{(J)}_{n(2)} = \frac{\|T_{nij}\|}{\|T^{(1,1)}_{n,ij}\| \; \|T^{(2,2)}_{n,ij}\|}$$

where $\|\cdot\|$ is the determinant of the matrix $((\cdot))$. S_n is analogous to the usual likelihood ratio test (cf. Anderson (1957), pp. 233) where instead of $T_{n,ij}$'s, the sample product moment correlations are used. We shall first consider the relation between $S^{(J)}_{n(2)}$ and $V^{(J)}_{n(2)}$.

THEOREM 4.1. If $\underset{\sim}{X}$ has a nonsingular distribution in the sense that $\underset{\sim}{H}$, defined by (3.4) and (3.5), is positive definite, then $\|T_n\|$ converges in probability to $\|\underset{\sim}{H}\|$ as $n \to \infty$, provided $E|T_{n,ii}|^{(2+\delta)/2} < \infty$ for some $\delta > 0$.

The proof of this theorem follows precisely as that of Theorem 4.2 of Puri and Sen (1966), and is therefore omitted.

THEOREM 4.2. Under $H^{(2)}_o$ defined in (2.2),

$$(4.23) \qquad |T_{n,ij}| = O_p(n^{-\frac{1}{2}}) \text{ for all } i=1,\ldots,\ell;\ j=\ell+1,\ldots,p .$$

The same result also holds under the permutational law $\mathcal{P}_n$.

Proof. First note that under $H^{(2)}_o$, $E(T_{n,ij}) = 0$, $i=1,\ldots,\ell$; $j=\ell+1,\ldots,p$. Furthermore, since $\frac{1}{n}\sum_{\alpha=1}^{n} J^2_{n(i)}(\alpha/n)$ is finite for all $i=1,\ldots,p$; and since by (3.1), $\frac{1}{n}\sum_{\alpha=1}^{n} J^2_{n(i)}(\alpha/n) \to 1$ for all $i=1,\ldots,p$, we find after a few simple steps that

$$(4.24) \qquad \mathrm{Var}_{H_o}(T_{n,ij}) = \frac{c}{n-1}, \quad c < \infty \text{ for all } i=1,\ldots,p;\ j=\ell+1,\ldots,p.$$

Hence using (4.19), (4.20) and (4.24) and Tchebyscheff's inequality we obtain the desired result.

THEOREM 4.3. *If* $|T_{n,ij}| = 0_p(n^{-\frac{1}{2}})$ *for all* $i=1,\ldots,\ell$; $j=\ell+1,\ldots,p$ *then*

(4.25) $$|V_{n(2)}^{(J)} + n \log S_{n(2)}^{(J)}| = 0_p(n^{-1}),$$ *provided* $\underset{\sim}{H}$ *is positive definite*.

Proof. By Laplace's expansion of the determinants (cf. Ferrar (1941))

(4.26) $$\|\underset{\sim}{T}_n\| = \sum_{1 \leq q_1 < q_2 < \cdots < q_\ell \leq p} \left\|T\begin{pmatrix}1 & 2 & \ldots & \ell \\ q_1 & q_2 & \cdots & q_\ell\end{pmatrix}\right\| \cdot \left\|T^*\begin{pmatrix}1 & 2 & \ldots & \ell \\ q_1 & q_2 & \cdots & q_\ell\end{pmatrix}\right\|$$

where $\left\|T\begin{pmatrix}1 & 2 & \ldots & \ell \\ q_1 & q_2 & \cdots & q_\ell\end{pmatrix}\right\|$ is the determinant consisting of the rows $1,2,\ldots,\ell$ and columns $q_1, q_2, \ldots, q_\ell$, and $\left\|T^*\begin{pmatrix}1 & 2 & \ldots & \ell \\ q_1 & q_2 & \cdots & q_\ell\end{pmatrix}\right\|$ is the complimentary determinant. By virtue of Theorem 4.2, we have the following results.

If only one of $q_1, q_2, \ldots, q_\ell$ is different from $1,2,\ldots,\ell$, then

(4.27) $$\left\|T\begin{pmatrix}1 & 2 & \ldots & \ell \\ q_1 & q_2 & \cdots & q_\ell\end{pmatrix}\right\| = 0_p(n^{-\frac{1}{2}}); \quad \left\|T^*\begin{pmatrix}1 & 2 & \ldots & \ell \\ q_1 & q_2 & \cdots & q_\ell\end{pmatrix}\right\| = 0_p(n^{-\frac{1}{2}}),$$

and,

if r of $q_1, q_2, \ldots, q_\ell$ are different from $1,2,\ldots,\ell$, then

(4.28) $$\left\|T\begin{pmatrix}1 & 2 & \ldots & \ell \\ q_1 & q_2 & \cdots & q_\ell\end{pmatrix}\right\| = 0_p(n^{-r/2}); \quad \left\|T^*\begin{pmatrix}1 & 2 & \ldots & \ell \\ q_1 & q_2 & \cdots & q_\ell\end{pmatrix}\right\| = 0_p(n^{-r/2})$$

for $r \geq 1$

Consequently, from (4.26), we obtain

$$\|T_{n,ij}\| = \|T_{n,ij}^{(1,1)}\|\,\|T_{n,ij}^{(2,2)}\|$$

$$(4.29) \qquad + \sum_{i=1}^{\ell} \sum_{i'=1}^{\ell} \sum_{j=\ell+1}^{p} \sum_{j'=\ell+1}^{p} T_{n,ij}\, T_{n,i'j'} \|T_{n,ii'}^{(1)}\|\,\|T_{n,jj'}^{(2)}\|$$

$$+ O_p(n^{-2})$$

where

$$(4.30) \qquad \|T_{n,ii'}^{(1)}\| = \text{cofactor of } T_{n,ii} \text{ in } T_{n,ij}^{(1,1)} \quad (\text{cf. } (4.18))$$

and

$$\|T_{n,jj'}^{(2)}\| = \text{cofactor of } T_{n,jj} \text{ in } T_{n,ij}^{(2,2)} \quad (\text{cf. } (4.18)).$$

Now by definition

$$T_{n(1)}^{ii'} = \|T_{n,ii'}^{(1)}\| / \|T_{n,ij}^{(1,1)}\|$$

(4.31) and

$$T_{n(2)}^{jj'} = \|T_{n,jj'}^{(2)}\| / \|T_{n,ij}^{(2,2)}\| \quad .$$

Hence, from (4.22), (4.29), (4.31) and Theorem 4.1 we obtain

$$(4.32) \qquad -n \log S_{n(2)}^{(J)} = n \sum_{i=1}^{\ell} \sum_{i'=1}^{\ell} \sum_{j=\ell+1}^{p} \sum_{j'=\ell+1}^{p} T_{n,ij}\, T_{n,i'j'}\, T_{n(1)}^{ii'}\, T_{n(2)}^{jj'} + O_p(n^{-1})$$

which proves the required result.

<u>THEOREM 4.4</u>. <u>Under the assumptions of Theorem 4.4, the permutation distribution of</u> $V_{n(2)}^{(J)}$ (or $-n \log S_{n(2)}^{(J)}$) <u>converges asymptotically to a chisquare distribution with</u> $\ell(p-\ell)$ <u>degrees of freedom</u>.

Proof. By virtue of (4.21) and Theorem 4.3 it suffices to show that any linear combination

$$U_n = n^{-\frac{1}{2}} \sum_{i=1}^{\ell} \sum_{j=\ell+1}^{p} d_{ij} T_{n,ij} \tag{4.33}$$

has asymptotically (under $\mathcal{P}_n$) a normal distribution in the limit with mean zero and finite variance. That the mean is zero and variance is finite follows by using (4.19) and (4.20). To prove the asymptotic normality of U_n, let us denote

$$Z_\alpha(\underset{\sim}{R}_n) = \sum_{i=1}^{\ell} \sum_{j=\ell+1}^{p} d_{ij} L^{(i)}_{n,R_{i\alpha}} L^{(j)}_{n,R_{j\alpha}}, \quad \alpha=1,\ldots,n \tag{4.34}$$

so that

$$U_n = n^{-\frac{1}{2}} \sum_{\alpha=1}^{n} Z_\alpha(\underset{\sim}{R}_n) . \tag{4.35}$$

Now for any $\underset{\sim}{E}_n \in S(\underset{\sim}{E}_n)$, defined in (4.11), $Z_\alpha(\underset{\sim}{R}_n)$ can assume n! equally likely values obtained by letting $\underset{\sim}{R}_n \in \underset{\sim}{\mathcal{R}}_n$. Essentially then by the same method of proof as in Wald-Wolfowitz-Noether theorem (cf. Fraser (1957), p. 237) it is easily seen that

$$E_{\mathcal{P}_n}\{U_n^r\}/\{E_{\mathcal{P}_n}(V_n^2)\}^{r/2} = \begin{cases} \dfrac{2k!}{k!2^k} + o(1) & \text{if } r=2k \\ o(1) & \text{if } r=2k+1 \end{cases} \tag{4.36}$$

for $k=0,1,2,\ldots$. Hence the theorem.

Thus for large samples, the permutation test based on $V^{(J)}_{n(2)}$ (or $-n \log S^{(J)}_{n(2)}$) reduces to the following rule:

$$\text{If } V^{(J)}_{n(2)} \text{ or } -n \log S^{(J)}_{n(2)} \begin{array}{ll} > \chi^2_{\ell(p-\ell),\alpha}, & \text{reject } H_o^{(1)} \\ < \chi^2_{\ell(p-\ell),\alpha}, & \text{accept } H_o^{(1)} \end{array} \tag{4.37}$$

where $\chi^2_{r,\alpha}$ is the $100(1-\alpha)\%$ point of a chi-square distribution with r degrees of freedom.

We will not discuss briefly the case of $q(q > 2)$ subsets. Here we partition $\underset{\sim}{X}_{\alpha}$ as in (2.1), $\underset{\sim}{H}$ as in (4.3), and $\underset{\sim}{T}_n$ accordingly. Thus

$$(4.38) \qquad \underset{\sim}{T}_n = \begin{pmatrix} ((T^{(1,1)}_{n,ij})), & \cdots, & ((T^{(1,q)}_{n,ij})) \\ \vdots & & \\ ((T^{(q,1)}_{n,ij})), & \cdots, & ((T^{(q,q)}_{n,ij})) \end{pmatrix}$$

where $((T^{(k,\ell)}_{n,ij}))$ is of the order $i_k \times i_\ell$; $k,\ell=1,\ldots,q$. Then, by analogy with parametric likelihood ratio test statistic, for the problem of testing

$$(4.39) \qquad H_o^{(q)}: \underset{\sim}{H}_{k,\ell} = \underset{\sim}{H}'_{\ell,k} = \underset{\sim}{0} \quad k\neq\ell=1,\ldots,p.$$

We consider the test statistic

$$(4.40) \qquad S^{(J)}_{n(q)} = \frac{\|T_{n,ij}\|}{\prod_{k=1}^{q} \|T^{(k,k)}_{n,ij}\|} .$$

Proceeding as in Theorem (4.3), we can show that $-n \log S^{(J)}_{n(q)}$ is asymptotically equivalent to a positive definite quadratic form in $\sum_{k<\ell=1}^{p} i_k\, i_\ell$ statistics $T_{n,k\ell}$ where $X_{k\alpha}$ and $X_{\ell\alpha}$ ($k<\ell=1,\ldots,p$; $\alpha=1,\ldots,n$) belong to two different subsets of variates. The permutation argument considered earlier readily extends to the present case, where we shall have $(n!)^{q-1}$ equally likely realizations. Further as a straight forward generalization of theorem 4.4, we shall arive at the conclusion that under the permutation model, $-n \log S^{(J)}_{n(q)}$ where $S^{(J)}_{n(q)}$ is defined in (4.39) has asymptotically a chi-square distribution with

$\sum_{k<\ell} i_k i_\ell$ degrees of freedom. Thus we have a test criterion very similar to (4.37) with the only difference that the degrees of freedom will be equal to $\sum_{k<\ell=1}^{p} i_k i_\ell$.

As a special case, we consider $q=p$. Here $i_1 = \ldots = i_p = 1$, and the problem reduces to that of testing the hypothesis of total independence of all the p variates, (cf. (2.3)). Here we may work with the test criterion

$$V_{n(p)}^{(J)} = (n-1) \sum_{i<j=1}^{p} T_{n,ij}^2 / T_{n,ii} T_{n,jj} \tag{4.41}$$

(which is asymptotically equivalent to $-\log S_{n(p)}^{(J)}$ where $S_{n(p)}^{(J)}$ is defined by (4.40) by putting $q=p$). In this case the permutation distribution of $V_{n(p)}^{(J)}$ coincides with the unconditional null distribution, and following the lines of theorem 4.4, it can be shown that under $H_o^{(p)}$ defined in (2.3), $V_{n(p)}^{(J)}$ has asymptotically a chi-square distribution with $p(p-1)/2$ degrees of freedom.

5. Asymptotic Normality of $\underset{\sim}{T}_n$ for arbitrary $F(\underset{\sim}{x})$. In this section we shall establish the joint asymptotic normality of the statistics $\{T_{n,ij};\ 1 \leq i < j \leq p\}$ defined by (3.9) for arbitrary $F(\underset{\sim}{x})$.

We shall make the following assumptions:

Assumption 5.1. $\lim_{n\to\infty} J_{n(i)}(u) = J_{(i)}(u)$ exists for $0 < u < 1$ and is not constant for $i=1,\ldots,p$.

Assumption 5.2.

$$\iint_{I_{n(i)} \times I_{n(j)}} \{J_{n(i)}[F_{n[i]}(x_i)]\, J_{n(j)}[F_{n[j]}(x_j)] - J_{(i)}[F_{[i]}(x_i)]\, J_{(j)}[F_{[j]}(x_j)]\}\, dF_{n[i,j]}(x_i, x_j) = o_p(n^{-\frac{1}{2}})$$

where

$$(5.1) \qquad I_{n(i)} = \{x_i: \ 0 < F_{n(i)}(x_i) < 1\}; \quad i=1,\ldots,p.$$

<u>Assumption 5.3</u>. $\qquad J_{n(i)}(1) = o(n^{\frac{1}{4}}); \quad i=1,\ldots,p.$

<u>Assumption 5.4</u>. $\qquad |J_{(i)}(u)| \leqq K[u(1-u)]^{-\alpha}, \quad 0 < \alpha < 1/8$

$$|J'_{(i)}(u)| \leqq K[u(1-u)]^{-1}$$

$$|J''_{(i)}(u)| \leqq K[u(1-u)]^{-2}$$

where $J'_{(i)}$ and $J''_{(i)}$ denote the first and second derivatives of $J_{(i)}$ respectively, K is some constant, and $i=1,\ldots,p$.

Finally, we shall denote

$$(5.2) \qquad \mu_{n(i,j)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} J_{(i)}\,[F_{(i)}(x_i)]\; J_{(j)}\,[F_{(j)}(x_j)]\; d\,F_{(i,j)}(x_i,x_j)$$

$$(5.3) \qquad n\,\sigma^2_{n(i,j)} = n\,\sigma_{n\{(i,j),(i,j)\}} = \mathrm{Var}\left[\sum_{\ell=1}^{3} U^{(\alpha)}_{(i,j);\ell}\right]$$

$$(5.4) \qquad n\,\sigma_{n\{(i,j),(r,s)\}} = \sum_{\ell=1}^{3}\sum_{\ell'=1}^{3} \mathrm{Cov}(U^{(\alpha)}_{(i,j);\ell},\; U^{(\alpha)}_{(r,s);\ell'})$$

where

$$(5.5) \qquad U^{(\alpha)}_{(r,s);\ell} = J_{(r)}\,[F_{[r]}(X_{r\alpha})]\; J_{(s)}\,[F_{[s]}(X_{s\alpha})] \qquad \text{if } \ell=1$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} [F_{1X_{r\alpha}}(x_r) - F_{[r]}(x_r)]\; J_{(s)}\,[F_{[s]}(x_s)]\; J'_{(r)}\,[F_{[r]}(x_r)]$$

$$d\,F_{[r,s]}(x_r,x_s), \quad \text{if } \ell=2$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [F_{1X_{s\alpha}}(x_s) - F_{[s]}(x_s)] \; J_{(r)}[F_{[r]}(x_r)] \; J'_{(s)}[F_{[s]}(x_s)] \; d\, F_{[r,s]}(x_r, x_s),$$

$$\text{if } \ell=3, \; r < s=1,\ldots,p$$

and, where

(5.6) $$F_{1X_{s\alpha}}(u) = 1 \text{ if } X_{s\alpha} \leqq u \text{ and zero otherwise.}$$

We shall now prove the following

THEOREM 5.1. Under the assumptions 5.1 to 5.4, the random vector with elements $n^{\frac{1}{2}}[(T_{n(i,j)} - \mu_{n(i,j)}),\; 1 \leqq i < j \leqq p]$ where $\mu_{n(i,j)}$ is given by (5.2) has a limiting normal distribution with mean vector zero, and covariance matrix $\underset{\sim}{\Sigma} = n((\sigma_{n\{(i,j),(r,s)\}}))$ given by (5.3) and (5.4) respectively.

Proof. We write

(5.7) $$J_{n(i)}[F_{n[i]}(x_i)]J_{n(j)}[F_{n[j]}(x_j)] = \{J_{n(i)}[F_{n[i]}(x_i)]J_{n(j)}[F_{n[j]}(x_j)]$$
$$- J_{(i)}[F_{n[i]}(x_i)]J_{(j)}[F_{n[j]}(x_j)]\} + J_{(i)}[F_{n[i]}(x_i)]$$
$$J_{(j)}[F_{n[j]}(x_j)]$$

and, (by Taylor's theorem)

(5.8) $$J_{(i)}[F_{n[i]}(x_i)] \; J_{(j)}[F_{n[j]}(x_j)] = J_{(i)}[F_{[i]}(x_i)] \; J_{(j)}[F_{[j]}(x_j)]$$
$$+ [F_{n[i]}(x_i) - F_{[i]}(x_i)] \; J'_{(i)}[F_{[i]}(x_i)] \; J_{(j)}[F_{[j]}(x_j)]$$
$$+ [F_{n[j]}(x_j) - F_{[j]}(x_j)] \; J'_{(j)}[F_{[j]}(x_j)] \; J_{(i)}[F_{[i]}(x_i)]$$
$$+ \tfrac{1}{2}[F_{n[i]}(x_i) - F_{[i]}(x_i)]^2 \; J''_{(i)}[\theta F_{n[i]}(x_i) + (1-\theta)F_{[i]}(x_i)] \; J_{(j)}[\theta F_{n[j]}(x_j)$$
$$+ (1-\theta)F_{[j]}(x_j)]$$

$$+ \tfrac{1}{2}[F_{n[j]}(x_j) - F_{[j]}(x_j)]^2 \, J''_{(j)}[\theta F_{n[j]}(x_j) + (1-\theta)F_{[j]}(x_j)] \, J_{(i)}[\theta F_{n[i]}(x_i) + (1-\theta)F_{[i]}(x_i)]$$

$$+ [F_{n[i]}(x_i) - F_{[i]}(x_i)][F_{n[j]}(x_j) - F_{[j]}(x_j)] \, J'_{(i)}[\theta F_{n[i]}(x_i) + (1-\theta)F_{[i]}(x_i)] \, J'_{(j)}[\theta F_{n[j]}(x_j) + (1-\theta)F_{[j]}(x_j)],$$

$$0 < \theta < 1$$

and

$$dF_{n[i,j]}(x_i, x_j) = d[F_{n[i,j]}(x_i, x_j) - F_{[i,j]}(x_i, x_j)] + dF_{[i,j]}(x_i, x_j). \tag{5.9}$$

Then, proceeding as in Bhuchongkul (1964), we can express $T_{n,ij}$ (cf. (3.9)) as

$$T_{n,ij} = \sum_{r=1}^{3} A_{rn}^{(i,j)} + \sum_{r=1}^{8} B_{rn}^{(i,j)} \tag{5.10}$$

where

$$A_{1N}^{(i,j)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} J_{(i)}[F_{[i]}(x_i)] \, J_{(j)}[F_{[j]}(x_j)] \, dF_{n[i,j]}(x_i, x_j) \tag{5.11}$$

$$A_{2N}^{(i,j)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [F_{n[i]}(x_i) - F_{[i]}(x_i)] J_{(j)}[F_{[j]}(x_j)] J'_{(i)}[F_{[i]}(x_i)] \, dF_{[i,j]}(x_i, x_j) \tag{5.12}$$

$$A_{3N}^{(i,j)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [F_{n[j]}(x_j) - F_{[j]}(x_j)] J_{(i)}[F_{[i]}(x_i)] J'_{(j)}[F_{[i]}(x_j)] \, dF_{[i,j]}(x_i, x_j) \tag{5.13}$$

and, the B terms are all $o_p(n^{-\frac{1}{2}})$; (cf. Bhuchongkul cited above).

The difference $n^{\frac{1}{2}}[T_{n,ij} - \sum_{r=1}^{3} A_{rn}^{(i,j)}]$ tends to zero in probability as n tends to infinity, and so, by a well known theorem [Cramer (1954 p. 299)], the vectors $n^{\frac{1}{2}}[T_{n,ij},\ 1 \leq i < j \leq p]$ and $n^{\frac{1}{2}}[\sum_{r=1}^{n} A_{rn}^{(i,j)},\ 1 \leq i < j \leq p]$ have the same

limiting distributions, if they have one at all. Thus, to prove the theorem, it suffices to show that for any real λ_{ij} $(1 \leq i < j \leq p)$, not all zero, $\sum_{i=1}^{p} \sum_{j=1}^{p} \lambda_{ij}(A_{1N}^{(i,j)} + A_{2N}^{(i,j)} + A_{3N}^{(i,j)})$ has normal distribution in the limit. Now $\sum_{r=1}^{3} A_{rn}^{(i,j)}$ can be expressed as

$$(5.14) \qquad \sum_{r=1}^{3} A_{rn}^{(i,j)} = \frac{1}{n} \sum_{\alpha=1}^{n} [U_{(i,j);1}^{(\alpha)} + U_{(i,j);2}^{(\alpha)} + U_{(i,j);3}^{(\alpha)}]$$

where the $U_{(i,j)}$'s are given by (5.5). Hence

$$(5.15) \qquad \sum_{\substack{i=1 \\ i<j}}^{p} \sum_{j=1}^{p} \lambda_{ij}\{\sum_{r=1}^{3} A_{rn}^{(i,j)}\} = \frac{1}{n} \sum_{\alpha=1}^{n} [\sum_{\substack{i=1 \\ i<j}}^{p} \sum_{j=1}^{p} \lambda_{ij}\{U_{(i,j);1}^{(\alpha)} + U_{(i,j);2}^{(\alpha)} + U_{(i,j);3}^{(\alpha)}\}]$$

The right hand side of (5.15) is the average of n independent and identically distributed random variables, each having mean $\sum_{\substack{i=1 \\ i<j}}^{p} \sum_{j=1}^{p} \lambda_{ij}\, \mu_{n(i,j)}$ and finite third moments. Tha asymptotic normality follows. Furthermore using (5.14), it is easy to check that the variance-covariance matrix $\underset{\sim}{\Sigma} = n((\sigma_{n\{(i,j),(r,s)\}}))$ is given by (5.3) and (5.4). The theorem follows.

The following theorem gives a simple sufficient condition under which the assumptions 5.1, 5.2 and 5.3 hold.

THEOREM 5.2. If $J_{n(i)}(\alpha/n)$ $(i=1,\ldots,p)$ is the expected value of the αth order statistic of a sample of size from a population whose cumulative distribution is the inverse function of $J_{(i)}$, $i=1,\ldots,p$ and if the assumption 5.4 of theorem 5.1 is satisfied, then the assumptions 5.1, 5.2 and 5.3 are also satisfied.

The proof of this theorem is analogous to that of theorem 2 of Bhuchongkul (1964) and is therefore omitted.

With the use of this theorem it is easy to verify that if $J_{n(i)}(\alpha/n)$ is the expected value of the αth order statistic of a sample of size n from (i) the standard normal distribution, (ii) the logistic distribution, (iii) the double exponential distributions, (iv) the exponential distribution, and (v) the uniform distribution, then the vector $n^{\frac{1}{2}}[(T_{n,ij} - \mu_{n(i,j)}), 1 \leq i < j \leq p]$ has a limiting normal distribution.

We shall use the results of theorem 5.1 in deriving large sample power properties of the statistics associated with the (i) tests of independence of $q \geq 2$ sets of variates, and (ii) tests for pair-wise independence. For the simplicity of presentation, we shall first consider the problem of testing independence of two sets of variates.

6. Testing Independence of Two Sets of Variates. Let the p component vector $\underset{\sim}{X}_\alpha$ be partitioned into two subvectors $\underset{\sim}{X}_\alpha^{(1)}$ and $\underset{\sim}{X}_\alpha^{(2)}$ as defined in (4.6).

Let the population dispersion matrix $\underset{\sim}{\Theta}$ defined in (3.4) and (3.5) be partitioned as we did the sample dispersion matrix $\underset{\sim}{T}_n$ defined in (3.10). Then we have

$$(6.1) \qquad \underset{\sim}{\Theta} = \begin{pmatrix} \underset{\sim}{\Theta}_{11} & \underset{\sim}{\Theta}_{12} \\ \underset{\sim}{\Theta}_{21} & \underset{\sim}{\Theta}_{22} \end{pmatrix}$$

Note that $\underset{\sim}{\Theta}_{11}$ is of order $\ell \times \ell$, $\underset{\sim}{\Theta}_{12}$ is of order $\ell \times m$. Then, for the problem of testing $H_o^{(2)}$

$$(6.2) \qquad H_o^{(2)}: \underset{\sim}{\Theta}_{12} = \underset{\sim}{\Theta}'_{21} = \underset{\sim}{0}$$

we propose to consider the statistic $S^{(J)}_{n(2)}$ defined by (4.22). The test consists in rejecting $H^{(2)}_{o}$ if $S^{(J)}_{n(2)}$ exceeds some predetermined number h_{ϵ}. We shall prove below that if $H^{(2)}_{o}$ is true, the statistic $-n \log S^{(J)}_{n(2)}$ has a limiting chi-square distribution with ℓm degrees of freedom as $n \to \infty$. This provides the user of this $S^{(J)}_{n(2)}$ test with a large sample approximation of the value of h_{ϵ} for any $0 < \epsilon < 1$.

We shall now study the asymptotic distribution of $S^{(J)}_{n(2)}$ assuming a sequence of alternative hypotheses $\{H^{(2)}_{n},\ n=1,2,\ldots\}$ which specifies that

$$(6.3) \qquad \Omega_{ij}(F_{[i]},F_{[j]}) = 1 + n^{-\frac{1}{2}}\,\omega_{ij}(F_{[i]},F_{[j]}); \quad i=1,\ldots,\ell,\ j=\ell+1,\ldots,p$$

where Ω_{ij} is defined by (3.13), ω_{ij} is some function of $(F_{[i]},F_{[j]})$, and $\omega_{ij} \not\equiv 0$ for all (i,j); $i=1,\ldots,\ell$; $j=\ell+1,\ldots,p$. It may be pointed out that the sequence of the alternative hypotheses $\{H^{(2)}_{n},\ n=1,2,\ldots\}$ defined in (6.3) implies that

$$(6.4) \qquad \underset{\sim}{\mathbb{H}}_{12} = \underset{\sim}{\mathbb{H}}'_{21} = n^{-\frac{1}{2}}\,\mathcal{B}_{12}$$

where $\mathcal{B}_{12} = ((\beta_{ij}))$ is an $\ell \times m$ matrix where elements are

$$(6.5) \qquad \beta_{ij} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_{[i]}(x)\,F_{[j]}(y)\,\omega_{ij}(F_{[i]},F_{[j]})J'_{(i)}[F_{[i]}(x)]J'_{(j)}[F_{[j]}(y)]\,dF_{[i]}(x)dF_{[j]}(y)$$

$$i=1,\ldots,\ell;\ j=\ell+1,\ldots,p.$$

The following lemma is an immediate consequence of the fact that $\operatorname{var}\{T_{n,ij}\}$ tends to zero as $n \to \infty$ for all $F \in \mathfrak{F}_p$.

<u>Lemma 6.1</u>. <u>Under the assumptions of theorem</u> 5.1,

(a) $\underset{\sim}{T}_n \to \underset{\sim}{\mathbb{H}}$ in probability as $n \to \infty$ for all $F \in \mathfrak{F}_p$.

(b) $\underset{\sim}{T}^{ii'}_{n(1)} \rightarrow \theta^{ii'}_{(1)}$ in probability as $n\rightarrow\infty$ for all $F \in \mathcal{F}_p$

$i, i'=1,\ldots,\ell$

(c) $T^{jj'}_{n(2)} \rightarrow \theta^{jj'}_{(2)}$ in probability as $n\rightarrow\infty$ for all $F \in \mathcal{F}_p$

$j, j'=\ell+1,\ldots,p.$

where

(6.6) $\theta^{ii'}_{(1)}$ and $\theta^{jj'}_{(2)}$ are the (i,i')th and (j,j')th elements of $\underset{\sim}{H}^{-1}_{11}$ and $\underset{\sim}{H}^{-1}_{22}$ respectively.

Lemma 6.2. If, for each $i < j = 1,\ldots,p$

(i) the conditions of theorem 5.1 are satisfied

(ii) the hypothesis $\{H_n^{(2)}\}$ defined by (6.3) is valid then, the matrix with elements $n^{\frac{1}{2}}[(T_{n,ij} - \mu_{n(i,j)}),\ i=1,\ldots,\ell;\ j=\ell+1,\ldots,p]$ has a limiting normal distribution with means zero and variance-covariance matrix $\Sigma^* = ((\sigma^*_{\{(i,j),(r,s)\}}))$ where

(6.7) $$\sigma^{2*}_{(i,j)} = \sigma^*_{\{(i,j),(i,j)\}} = 1 \qquad \text{if} \quad i=1,\ldots,\ell;\ j=\ell+1,\ldots,p$$

(6.8) $$\sigma^*_{\{(i,j),(r,s)\}} = \theta_{ir}\,\theta_{js} \qquad i,r=1,\ldots,\ell;\ j,s=\ell+1,\ldots,p.$$

(note that $\theta_{ii} = 1$).

The proof of this lemma is an immediate consequence of theorem 5.1, (6.3) and (3.1), and is therefore omitted.

THEOREM 6.1. Under the assumptions of lemma 6.2, the limiting distribution of $-n \log S_{n(2)}^{(J)}$ is non central chi-square with ℓ m degrees of freedom and noncentrality parameter

$$(6.10) \quad \Delta_{S_{n(2)}}(J) = \sum_{i=1}^{\ell} \sum_{i'=1}^{\ell} \sum_{j=\ell+1}^{p} \sum_{j'=\ell+1}^{p} \beta_{ij} \beta_{i'j'} \theta_{(1)}^{ii'} \theta_{(2)}^{jj'}$$

Proof. Under the assumed conditions $n^{\frac{1}{2}} T_{n,ij}$ is bounded in probability for all $i=1,\ldots,\ell$; $j=\ell+1,\ldots,p$: Hence applying Laplace expansion for the determinant $\|\underset{\sim}{T}_n\|$ and proceeding exactly as in theorem 4.3, we obtain

$$(6.11) \quad -n \log S_{n(2)}^{(J)} = n \sum_{i=1}^{\ell} \sum_{i'=1}^{\ell} \sum_{j=\ell+1}^{p} \sum_{j'=\ell+1}^{p} T_{n,ij} T_{n,i'j'} T_{n(1)}^{ii'} T_{n(2)}^{jj'} + O_p(n^{-1}).$$

Using lemma 6.1, we notice that $-n \log S_{n(2)}^{(J)}$ is asymptotically equivalent to V_n^* where

$$(6.12) \quad V_n^* = n \sum_{i=1}^{\ell} \sum_{i'=1}^{\ell} \sum_{j=\ell+1}^{p} \sum_{j'=\ell+1}^{p} T_{n,ij} T_{n,i'j'} \theta_{(1)}^{ii'} \theta_{(2)}^{jj'}$$

where $\theta_{(1)}^{ii'}$ and $\theta_{(2)}^{jj'}$ are defined in (6.6). The result now follows by an application of lemma 6.2.

We shall consider the special forms of the function $\Delta_{S_{n(2)}}(J)$ for suitable choices of the function J in section 8 where we consider the asymptotic properties of the $S_{n(2)}^{(J)}$ tests in relation to their parametric competitors based on the likelihood ratio test [cf. Anderson (1947), p. 233].

In the passing we may remark that forthe problem of testing independence of q sets of variates against the sequence of alternatives $\{H_n^{(q)}, n=1,2,\ldots\}$ which specify that $\Omega_{ij}^{(n)}$ is of the form (6.3) for all i, j belonging to different sets, it can be shown by the use of theorem 5.1 that $-n \log S_{n(q)}^{(J)}$ (cf. (4.40))

has asymptotically a non-central chi-square distribution with $\sum_{k<\ell=1}^{p} i_k\, i_\ell$ degrees of freedom. The details are omitted because of the intended brevity of the presentation.

In a relatively simple case, when q=p, the statistic (4.41) has under the sequence of alternatives (6.3), asymptotically, a non-central chi-square distribution with p(p-1)/2 degrees of freedom and non-centrality parameter $\Delta_{S_{n(p)}}(J)$ given by

$$\Delta_{S_{n(p)}}(J) = \sum_{i<j=1}^{p} \beta_{ij}^2 \tag{6.13}$$

where β_{ij} is defined in (6.5).

7. Pair wise Independence. In view of the fact that the pair wise independence does not in general imply total independence, we shall consider this problem seperately in this section. To be precise we consider the problem of testing

$$H_o^*:\ F_{[i,j]}(x_i, x_j) = F_{[i]}(x_i) \cdot F_{[j]}(x_j) \text{ for all pairs } (i,j) \tag{7.1}$$

against the sequence of alternatives $\{H_n^*,\ n=1,2\ldots\}$ which specifies that

$$F_{ij}(x_i, x_j) = F_i(x_i) \cdot F_j(x_j)\ \{1 + n^{-\frac{1}{2}}\ \omega_{ij}(F_i, F_j)\} \tag{7.2}$$

where $\omega_{ij}(F_i, F_j)$ is not identically equal to zero (a.e.), for at least one pair (i,j); $1 \leq i < j \leq p$. The test-statistic proposed for this problem is based on the statistic

$$\mathcal{L}_{n(J)} = n\ \underset{\sim}{T}_n^*\ \hat{\Gamma}^{-1}\ \underset{\sim}{T}_n^{*\prime} \tag{7.3}$$

where

$$\underset{\sim}{T}_n^* = (T_{n,12}, T_{n,13}, \ldots, T_{n,(p-1,p)}) \tag{7.4}$$

and $\hat{\Gamma}^{-1}$ is a consistent estimator of Γ^{-1} which is the inverse of the co-variance matrix of $n^{\frac{1}{2}} T_n^*$.

THEOREM 7.1. If, for each $i < j = 1,\ldots,p$

(i) the conditions of theorem 5.1 are satisfied

(ii) the hypothesis H_n^* defined by (7.2) is valid

then, the statistic $\mathscr{L}_{n(j)}$ defined by (7.3) has asymptotically a non-central chi square distribution with $p(p-1)/2$ degrees of freedom, and non centrality parameter $\Delta_{\mathscr{L}_{n(J)}}$ defined as

(7.5) $$\Delta_{\mathscr{L}_{n(J)}} = \underset{\sim}{\Delta}\ \Gamma^{-1}\ \underset{\sim}{\Delta}$$

where

(7.6) $$\underset{\sim}{\Delta} = (\beta_{12}, \beta_{13}, \ldots, \beta_{p-1,p})$$

(The β_{ij}'s are defined in (6.5)).

Proof. The proof of this theorem is an immediate consequence of the fact that under the given assumptions, the random vector $\sqrt{n}\ \underset{\sim}{T}_n^*$ defined by (7.4) has a limiting normal distribution with mean vector $\underset{\sim}{\Delta}$ given by (7.6) and the co-variance matrix $\Gamma = ((\tau_{\{(i,j),(r,s)\}}))$ where

(7.7) $$\tau_{\{(i,j),(r,s)\}} = \begin{cases} 1, & \text{if } i=r,\ j=s \\ E_o[J_{(i)}(F_i(x_i))J_{(j)}(F_j(x_j))J_{(r)}(F_{(r)}(x_r))J_{(s)}(F_s(x_s)] & \text{if } i\neq r,\ j\neq s \\ E_o[J^2_{(i)}(F_i(x_i))J_{(j)}(F_j(x_j))J_{(s)}(F_s(x_s))] & \text{if } i=r,\ j\neq s \\ E_o[J_{(i)}(F_i(x_i))J^2_{(j)}(F_j(x_j))J_{(r)}(F_r(x_r))] & \text{if } i\neq r,\ j=s \\ E_o[J^2_{(i)}(F_i(x_i))J_{(j)}(F_j(x_j))J_{(r)}(F_r(x_r))] & \text{if } i=s,\ j\neq r. \end{cases}$$

and, the fact that $\hat{\Gamma}^{-1}$ is a consistent estimator of Γ^{-1}.

From theorem 7.1, it is clear that any consistent estimator of $\underset{\sim}{\Gamma}^{-1}$ will preserve the asymptotic distribution of the statistic $\mathcal{L}_{n(J)}$. A natural consistent estimator of (7.7) may be obtained readily by replacing $J_{(i)}(F_i(x_i))$ by $J_{n(i)}(F_{n(i)}(x_{i\alpha}))$ for all i, and the expectation by sample averages.

8. Asymptotic Relative Efficiency. In this section we shall make large sample power comparisons between the $S_{n(2)}^{(J)}$ tests, $\mathcal{L}_n^{(J)}$ tests, and their parametric competitors based on the likelihood ratio test. First we consider the interesting particular cases of the $S_{n(2)}^{(J)}$ and $\mathcal{L}_n^{(J)}$ tests.

(a) Special cases of the $S_{n(2)}^{(J)}$ tests.

a(i) Let $J_{(i)}(u) = \Phi^{-1}(u)$ where Φ is the standard normal distribution function. The $-n \log S_{n(2)}^{(J)}$ test then reduces to the normal scores $-n \log S_{n(2)}^{(\Phi)}$ test. The non-centrality parameter in this case is $\Delta_{S_{n(2)}}(\Phi)$ where

$$(8.1) \qquad \Delta_{S_{n(2)}}(\Phi) = \sum_{i=1}^{\ell} \sum_{i'=1}^{\ell} \sum_{j=\ell+1}^{p} \sum_{j'=\ell+1}^{p} \beta_{ij}\, \beta_{i'j'}\, \theta_{(1)}^{ii'}\, \theta_{(2)}^{jj'}$$

where

$$(8.2) \quad \beta_{ij} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F_{[i]}(x) F_{[j]}(y) \omega_{ij}(F_{[i]}, F_{[j]}) \frac{1}{\phi[\Phi^{-1}(F_{[i]}(x)]} \cdot \frac{1}{\phi[\Phi^{-1}[F_{[j]}(x)]} dF_{[i]}(x)\, dF_{[j]}(y) \qquad \text{(cf. (6.5))}$$

and

(8.3) $\theta^{ii'}$, $\theta^{jj'}$ are the (i,i')th and (j,j')th terms of

$$\Theta_{11}^{-1} = ((\theta_{ij}))_{\ell\times\ell}^{-1} \text{ and } \Theta_{22}^{-1} = ((\theta_{ij}))_{m\times m}^{-1} \text{ respectively, and where}$$

$$(8.4) \qquad \theta_{ij} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi^{-1}[F_{[i]}(x)]\, \Phi^{-1}[F_{[j]}(y)]\, dF_{[i,j]}(x,y).$$

In an important case, when $F(\underset{\sim}{x})$ is $N(\underset{\sim}{\mu}, \underset{\sim}{\Sigma})$

where

(8.5) $$\underset{\sim}{\Sigma} = ((\rho_{ij}^{(n)}))_{p\times p}\ ; \quad \rho_{ij}^{(n)} = \frac{1}{\sqrt{n}}\lambda_{ij}, \quad i=1,\ldots,\ell\ ;\ j=\ell+1,\ldots,p \quad \text{or} \quad i=\ell+1,\ldots,p;\ j=1,\ldots,\ell$$

and $\rho_{ij}^{(n)} = \rho_{ij}$ otherwise.

We have

(8.6) $$\omega_{ij}(F_{[i]},F_{[j]}) = \sqrt{n}\left[\frac{\Phi_{ij}(x,y,\rho_{ij}^{(n)})}{\Phi_i(x)\ \Phi_j(y)} - 1\right]$$

where $\Phi_{ij}(x,y,\rho_{ij}^{(n)})$ is the normal cdf of (x_i,x_j) where X_i is N(0,1), X_j is N(0,1) and $\rho_{ij}^{(n)}$ is the correlation coefficient between X_i and X_j. Expanding $\Phi_{ij}(x,y,\rho_{ij}^{(n)})$ about $\rho_{ij}^{(n)} = 0$, and neglecting the terms of order $o(n^{-\frac{1}{2}})$, we obtain (also see Sibuya (1959))

(8.7) $$\omega_{ij}(F_{[i]},F_{[j]}) = \lambda_{ij}\frac{\phi_i(x)\ \phi_j(y)}{\Phi_i(x)\ \Phi_j(y)}$$

where ϕ_i is the density of Φ_i; i=1,...,p. Hence, when F(x) is normal $N(\underset{\sim}{\mu},\underset{\sim}{\Sigma})$

(8.8) $$\beta_{ij} = \lambda_{ij};\quad i=1,\ldots,\ell\ ;\ j=\ell+1,\ldots,p.$$

(8.9) $$\theta_{ij} = \rho_{ij};\quad i=1,\ldots,\ell\ ;\ j=1,\ldots,\ell\ ;\ \text{or}\ i=\ell+1,\ldots,p;\ j=\ell+1,\ldots,p.$$

Hence from (8.1), in case $F(\underset{\sim}{x})$ is normal

(8.10) $$\Delta_{S_{n(2)}}(\Phi) = \sum_{i=1}^{\ell}\ \sum_{i'=1}^{\ell}\ \sum_{j=\ell+1}^{p}\ \sum_{j'=\ell+1}^{p} \lambda_{ij}\ \lambda_{i'j'}\ \rho_{(1)}^{ii'}\ \rho_{(2)}^{jj'}$$

where

(8.11) $\rho_{(1)}^{ii'}$ and $\rho_{(2)}^{jj'}$ are the (i,i')th and (j,j')th terms of

$$\mathcal{P}_{11}^{-1} = ((\rho_{ij}))^{-1}_{i,j=1,\ldots,\ell}\ ;\quad \mathcal{P}_{22}^{-1} = ((\rho_{ij}))^{-1}_{i,j=\ell+1,\ldots,p}$$

a(ii) Let $J_{(i)}(u) = \sqrt{12}\,(u-\frac{1}{2})$, then the $-n \log S^{(J)}_{n(2)}$ test reduces to the rank sum statistic $-n \log S^{(R)}_{n(2)}$ which is a multivariate analogue of the Spearman's rank correlation statistic [cf. Kendall (1955)]. The non-centrality parameter in this case reduces to $\Delta_{S^{(R)}_{n(2)}}$, where

$$(8.12)\qquad \Delta_{S^{(R)}_{n(2)}} = \sum_{i=1}^{\ell} \sum_{i'=1}^{\ell} \sum_{j=\ell+1}^{p} \sum_{j'=\ell+1}^{p} \beta_{ij}\, \beta_{i'j'}\, \theta^{ii'}_{(1)}\, \theta^{jj'}_{(2)}$$

where

$$(8.13)\qquad \beta_{ij} = 12 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_{[i]}(x) F_{[j]}(y)\, \omega_{ij}(F_{[i]}, F_{[j]})\, dF_{[i]}(x)\, dF_{[j]}(y)$$

and

$\theta^{ii'}_{(1)}$, $\theta^{jj'}_{(2)}$ are the (i,i')th and (j,j')th terms of $\Theta^{-1}_{11} = ((\theta_{ij}))_{i,j=1,\ldots,\ell}$ and $\Theta^{-1}_{22} = ((\theta_{ij}))_{i,j=\ell+1,\ldots,p}$, respectively and, where

$$(8.14)\qquad \theta_{ij} = 12 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (F_i(x) - \tfrac{1}{2})(F_j(y) - \tfrac{1}{2})\, dF_{[i,j]}(x,y)$$

In case $F(\underset{\sim}{x})$ is $N(\underset{\sim}{\mu}, \underset{\sim}{\Sigma})$ where $\underset{\sim}{\Sigma} = ((\rho^{(n)}_{ij}))$, $\rho^{(n)}_{ij} = \rho_{ij}$ for $i,j=1,\ldots,\ell$ or $i,j=\ell+1,\ldots,p$ and $\rho^{(n)}_{ij} = \frac{1}{\sqrt{n}} \lambda_{ij}$ for $i=\ell,\ldots,\ell$; $j=\ell+1,\ldots,p$ or $i=\ell+1,\ldots,p$; $j=1,\ldots,\ell$

the non-centrality parameter $\Delta_{S^{(R)}_{n(2)}}$ reduces to

$$(8.15)\qquad \Delta_{S^{(R)}_{n(2)}} = \frac{9}{\pi^2} \sum_{i=1}^{\ell} \sum_{i'=1}^{\ell} \sum_{j=\ell+1}^{p} \sum_{j'=\ell+1}^{p} \lambda_{ij}\, \lambda_{i'j'}\, \rho^{*ii'}_{(1)}\, \rho^{*jj'}_{(2)}$$

where

(8.16) $\rho^{*ii'}_{(1)}$ and $\rho^{*jj'}_{(2)}$ are the (i,i')th and (j,j')th terms of

$$\rho_{11}^{-1*} = \left(\left(\frac{6}{\pi}\operatorname{Sin}^{-1}\frac{\rho_{ij}}{2}\right)\right)^{-1}_{i,j=1,\ldots,\ell} \quad \text{and} \quad \rho_{22}^{-1*} = \left(\left(\frac{6}{\pi}\operatorname{Sin}^{-1}\frac{\rho_{ij}}{2}\right)\right)^{-1}_{i,j=\ell+1,\ldots,p}.$$

respectively.

(b) Special Cases of $\mathcal{L}_n^{(J)}$ tests.

8(b.1) Let $J_{(i)}(u) = \Phi^{-1}(u)$. Then the $\mathcal{L}_n(J)$ reduces to the normal scores $\mathcal{L}_n(\Phi)$ test. The non-centrality parameter is then (cf. (7.5))

$$(8.17) \qquad \Delta_{\mathcal{L}_n}(\Phi) = \underset{\sim}{\lambda}\,\underset{\sim}{\Gamma}^{-1}\,\underset{\sim}{\lambda}'$$

where

$$\underset{\sim}{\lambda} = (\lambda_{12}, \lambda_{13}, \ldots, \lambda_{p-1,p}), \text{ and}$$

$$\underset{\sim}{\Gamma}^{-1} = ((\tau_{\{(i,j),(r,s)\}})) \text{ is given by (7.7).}$$

In a special case when $F(x)$ is $N(\underset{\sim}{\mu}, \underset{\sim}{\Sigma})$ where $\underset{\sim}{\Sigma} = ((\rho_{ij}^{(n)}))$, pairwise independence is equivalent to total independence and, hence from (8.7),

$$(8.18) \qquad \Delta_{\mathcal{L}_n}(\Phi) = \underset{\sim}{\lambda}\,\underset{\sim}{\lambda}'$$

(8b.2) Let $J_{(i)}(u) = \sqrt{12}(u-\frac{1}{2})$, then the $\mathcal{L}_n(J)$ test becomes the rank sum $\mathcal{L}_n(R)$ test. The non-centrality parameter, for the case when $F(\underset{\sim}{x})$ is $N(\underset{\sim}{\mu}, \underset{\sim}{\Sigma})$, is

$$(8.19) \qquad \Delta_{\mathcal{L}_n}(R) = \frac{9}{\pi^2}\,\underset{\sim}{\lambda}\,\underset{\sim}{\lambda}'$$

(c) Parametric Theory. In the parametric theory a commonly used test for the hypothesis $H_o^{(2)}$ is based on the statistic [see Anderson (1957), p. 242)]

$$(8.20) \qquad V^* = |A|/|A_{11}|\cdot|A_{22}$$

where

(8.21) $$A = \sum_{\alpha=1}^{n} (\underset{\sim}{x}_\alpha - \bar{\underset{\sim}{x}})'(\underset{\sim}{x}_\alpha - \bar{\underset{\sim}{x}}) = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

(8.22) $$A_{ij} = \sum_{\alpha=1}^{n} (\underset{\sim}{x}_\alpha^{(i)} - \bar{\underset{\sim}{x}}^{(i)})'(\underset{\sim}{x}_\alpha^{(j)} - \bar{\underset{\sim}{x}}^{(j)}); \quad i=1,2$$

(8.23) $$\bar{\underset{\sim}{x}}^{(i)} = \sum_{\alpha=1}^{n} \underset{\sim}{x}_\alpha^{(i)}/n$$

It is easy to see, by an application of a theorem due to Wald (1943) that under the sequence of alternatives of the type $H_n^{(2)}$, $-n \log V^*$ has asymptotically a non-central chi-square distribution with ℓm degrees of freedom and non-centrality parameter $\Delta_{V_n^*}$ given by

(8.24) $$\Delta_{V_n^*} = \sum_{i=1}^{\ell} \sum_{i'=1}^{\ell} \sum_{j=\ell+1}^{p} \sum_{j'=\ell+1}^{p} \lambda_{ij} \lambda_{i'j'} \rho_{(1)}^{ii'} \rho_{(2)}^{jj'}$$

where $\rho_{(1)}^{ii'}$ and $\rho_{(2)}^{jj'}$ are defined in (8.11).

Furthermore, as mentioned earlier, since under the normal theory model, pair wise independence is equivalent to total independence, therefore the tests for $H_o^{(p)}$ and H_o^* are based on the same statistic, viz.

(8.25) $$U^* = \frac{|A|}{\prod_{i=1}^{p} a_{ii}}$$

where $A = ((a_{ij}))_{i,j=1,\ldots,p}$ is defined in (8.21), and

(8.26) $$a_{ij} = \sum_{\alpha=1}^{n} (x_{i\alpha} - \bar{x}_i)(x_{j\alpha} - \bar{x}_j); \quad \bar{x}_i = \sum_{\alpha=1}^{n} x_{i\alpha}/n.$$

It turns out, (again by an application of Wald's theorem (1943)) that $-n \log U^*$

has under the sequence of alternatives (7.2) (which in the case of normal theory model imply $\rho_{ij}^{(u)} = \frac{1}{\sqrt{n}} \lambda_{ij}$ for all i, j=1, ..., p) asymptotically a non-central chi-square distribution with p(p-1)/2 degrees of freedom and non-centrality parameter

$$(8.27) \qquad \Delta_U^* = \underset{\sim}{\lambda}\, \underset{\sim}{\lambda}'$$

where $\underset{\sim}{\lambda}$ is defined in (8.17).

Now it is well known [cf. Puri and Sen (1966)] that if the two statistics have, under the alternative hypothesis, non-central chi-square distributions with the same number of degrees of freedom, then the asymptotic relative efficiency of one test with respect to the other test is equal to the ratio of their non-centrality parameters after the alternatives have been set equal.

Hence, denoting $e_{[T, T^*]}$, as the asymptotic efficiency of a test T relative to T*, we have

$$(8.28) \qquad e_{[S_{n(2)}^{(\Phi)},\ V^*]} = \Delta_{S_{n(2)}^{(\Phi)}} / \Delta_{V^*}$$

where $\Delta_{S_{n(2)}^{(\Phi)}}$ and Δ_{V^*} are defined in (8.1) and (8.29), respectively.

$$(8.29) \qquad e_{[S_{n(2)}^{(R)},\ V^*]} = \Delta_{S_{n(2)}^{(R)}} / \Delta_{V^*}$$

where $\Delta_{S_{n(2)}^{(R)}}$, and Δ_V^* are defined in (8.12) and (8.29), respectively.

In particular, where F(x) is $N(\underset{\sim}{\mu},\ ((\rho_{ij}^{(u)})))$, it is easy to notice from (8.10), (8.15) and (8.29) that

$$(8.30) \qquad e_{[S_{n(2)}^{(\Phi)},\ V^*]} = 1$$

$$(8.31) \qquad e_{[S_{n(2)}^{(R)}, V^*]} = \frac{9}{\pi^2} \frac{\sum_{i=1}^{\ell} \sum_{i'=1}^{\ell} \sum_{j=\ell+1}^{p} \sum_{j'=\ell+1}^{p} \lambda_{ij} \lambda_{i'j'} \rho^{*ii'} \rho^{*jj'}}{\sum_{i=1}^{\ell} \sum_{i'=1}^{\ell} \sum_{j=\ell+1}^{p} \sum_{j'=\ell+1}^{p} \lambda_{ij} \lambda_{i'j'} \rho^{ii'} \rho^{jj'}}$$

where $\rho^{*ii'}_{(1)}$, $\rho^{*jj'}_{(2)}$, $\rho^{ii'}_{(1)}$ and $\rho^{jj'}_{(2)}$ are defined in (8.11) and (8.16) respectively.

From (8.30) and (8.31) we note that whereas in the case of normal distributions, the property of the bivariate normal scores test of independence (cf. Bhuchongkul (1964)) relative to the likelihood ratio test is preserved in the multivariate case the same is not true in the case of multivariate rank sum $S_{n(2)}^{(R)}$ test. However, for special cases of the type dealt in Sen and Puri (1966) one may consider finding the bounds of (8.31). We do not pursue this problem in this paper.

Finally, it is easy to notice that under the normal theory model, in the case of pair wise independence (or total independence),

$$(8.32) \qquad e_{\mathcal{L}_n}(\Phi), U^* = 1$$

$$(8.33) \qquad e_{\mathcal{L}_n}(R), U^* = \frac{9}{\pi^2}$$

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